Design of IIR Filters from Analog Filters

- Motivation and Preliminaries
- IIR Filter Design by Approximation of Derivatives
- IIR Filter Design by Impulse Invariance
- IIR Filter Design by the Bilinear Transformation

Design of IIR Filters from Analog Filters

Objective: Design digital IIR filters using known analog filter results

- Analog filter design is mature and well understood

Recall: Analog systems are defined by

$$H_a(s) \;=\; \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} = \int_{-\infty}^{\infty} h(t) e^{-st} dt \qquad \text{[Laplace Trans.]}$$

Or, noting that s is the differentiation operator,

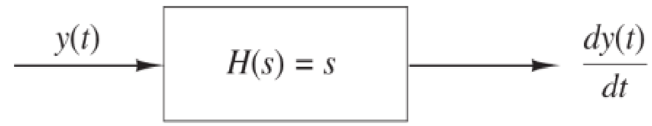
$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \qquad \text{[Differential Equation]}$$

Approach: Map analog filters (s–plane) to digital filters (z–plane)

- Frequency Mapping: s–domain $j\Omega$ axis $\rightarrow$ z–domain unit circle
- Stability Mapping: s–domain LHP $\rightarrow$ inside z–domain unit circle

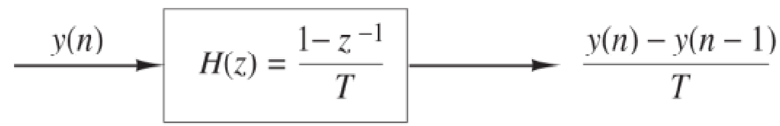
IIR Filter Design by Approximation of Derivatives

Approach: Since s it is the differentiation operator, define a discreet–time approximation



Continuous–time differentiator

Utilize the backward difference as a derivative approximation



Discrete–time backward difference difference differentiator

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT)-y(nT-T)}{T} = \frac{y(n)-y(n-1)}{T}$$

Which has the system function $H(z) = \frac{1-z^{-1}}{T}$. Equating the operations

$$s = \frac{1-z^{-1}}{T} \qquad \text{or} \qquad H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

Question: What s–plane $\leftrightarrow$ z–plane mapping does this perform?

$$s = \frac{1 - z^{-1}}{T} \quad \Rightarrow \quad z = \frac{1}{1 - sT}$$

Restrict s to the frequency axis, i.e., $s = \sigma + j\Omega$, with $\sigma = 0$

$$\begin{aligned} z &= \frac{1}{1 - j\Omega T} = \frac{1}{1 - j\Omega T}\left(\frac{1 + j\Omega T}{1 + j\Omega T}\right) \\ &= \frac{1}{1 + \Omega^2 T^2} + j\frac{\Omega T}{1 + \Omega^2 T^2} \end{aligned}$$

Check limiting points

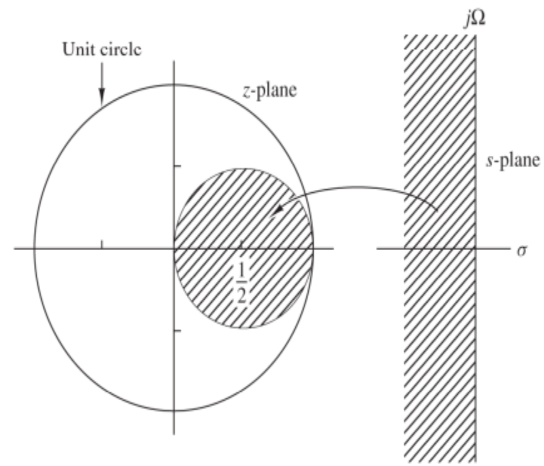
$$\lim_{\Omega \to 0} \frac{1}{1 - j\Omega T} \Rightarrow z \to 1 \qquad \text{[DC to DC]}$$

and

$$\lim_{\Omega \to \infty} \frac{1}{1 - j\Omega T} \Rightarrow z \to 0 \qquad \text{[HF to ??]}$$

To see general mapping, examine the point $z - \frac{1}{2}$ (with $s = j\Omega$)

$$
\begin{aligned}
z - \frac{1}{2} &= \frac{1}{1 - j\Omega T} - \frac{1}{2} \\
&= \frac{2 - (1 - j\Omega T)}{2(1 - j\Omega T)} \\
&= \frac{1 + j\Omega T}{2(1 - j\Omega T)} \\
\Rightarrow \left| z - \frac{1}{2} \right| &= \frac{|1 + j\Omega T|}{2|1 - j\Omega T|} \\
&= \frac{1}{2}
\end{aligned}
$$



$z = \frac{1}{1 - j\Omega T}$ induced s–plane to z–plane mapping

Observations:

- Mapping yields stable filters – LHP of the s–plane $\rightarrow$ inside the z–plane unit circle
- Image inside the unit circle is in right half $\Rightarrow$ *No high pass filters*

Example

Convert the analog bandpass filter

$$H_a(s) = \frac{1}{(s+0.1)^2 + 9}$$

into a digital filter using backward difference derivative approximation.

Utilizing $s = \frac{1-z^{-1}}{T}$ in the above

$$\begin{aligned} H(z) &= \frac{1}{\left(\frac{1-z^{-1}}{T} + 0.1\right)^2 + 9} \\ &= \frac{T^2/(1+0.2T+9.01T^2)}{1 - \frac{2(1+0.1T)}{1+0.2T+9.01T^2} z^{-1} + \frac{1}{1+0.2T+9.01T^2} z^{-2}} \end{aligned}$$

Given T, this reduces a simple system

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

IIR Filter Design by Impulse Invariance

Objective: Design a discrete–time IIR filter by sampling the impulse response of a continuous filter

Thus if $h_a(t)$ is a continuous–time impulse response, set

$$h(n) = h_a(nT), \quad n = 0, 1, \ldots$$

Consider the sampling induced frequency domain relation

$$\begin{aligned} H_a(s) &= \int_0^\infty h_a(t) e^{-st} dt \qquad \text{[Laplace Trans.]} \\ &= \sum_{n=0}^{\infty} h_a(nT) e^{-snT} \\ &= \sum_{n=0}^{\infty} h(n) e^{-snT} = H(z)|_{z=e^{sT}} \end{aligned}$$

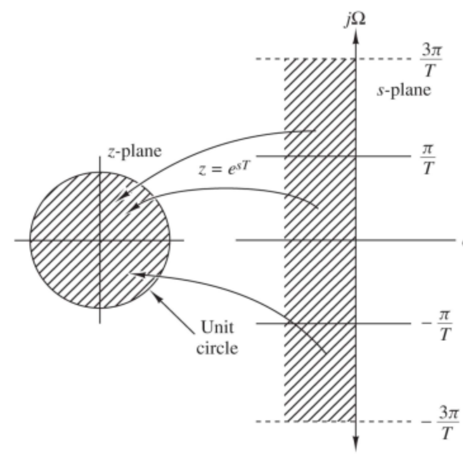
Result: This defines the the s–plane to z–plane mapping $z = e^{sT}$

Question: What s–plane $\leftrightarrow$ z–plane mapping does this perform?

Utilize $z = re^{j\omega}$ and $s = \sigma + j\Omega$ representations in the mapping

$$\begin{aligned} z &= e^{sT} \\ re^{j\omega} &= e^{\sigma T} e^{j\Omega T} \end{aligned}$$

$$\Rightarrow r = e^{\sigma T} \quad \text{and} \quad \omega = \Omega T$$



$z = e^{sT}$ induced s–plane to z–plane mapping

Observations:

- Mapping introduces aliasing (i.e., s and $(s + jk2\pi/T) \rightarrow$ same z)
- Zeros and poles don't follow the same mapping

Consider the pole mapping for arbitrary $H_a(s)$

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

Note $H_a(s)$ has N poles at $p_1, p_2, \ldots, p_N$

Since $H_a(s)$ is in partial fraction form, the impulse response is

$$
\begin{aligned}
h_a(t) &= \sum_{k=1}^{N} c_k e^{p_k t}, \quad t \geq 0 \\
\Rightarrow h(n) &= \sum_{k=1}^{N} c_k e^{p_k nT} \qquad \text{[after sampling]} \\
\Rightarrow H(z) &= \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k nT} \right) z^{-n} \qquad [z\text{–transform}] \\
&= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1} \right)^n \\
&= \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}
\end{aligned}
$$

Result: $H(z)$ has N poles at $e^{p_1 T}, e^{p_2 T}, \ldots e^{p_N T}$, i.e., the s–domain poles $p_1, p_2, \ldots p_N$ are mapped to the z–domain according to $z = e^{sT}$

Note: Zero mapping is filter dependent – show by example

Example

Convert the analog bandpass filter

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital filter using impulse invariance.

Note that $H_a(s)$ has the following zero and poles

$$\text{zero: } s = -0.1 \qquad \text{poles: } p_k = -0.1 \pm j3$$

and partial fraction expansion

$$H(s) = \frac{1/2}{s - (-0.1 + 3j)} + \frac{1/2}{s - (-0.1 - 3j)}$$

Mapping the poles according to $z = e^{sT}$ yields

$p_1 = -0.1 + j3 \rightarrow e^{-0.1T} e^{j3T}$ and $p_2 = -0.1 - j3 \rightarrow e^{-0.1T} e^{-j3T}$

Thus the poles in the z–domain are $p_k = e^{-0.1T} e^{\pm j3T}$. Mapping

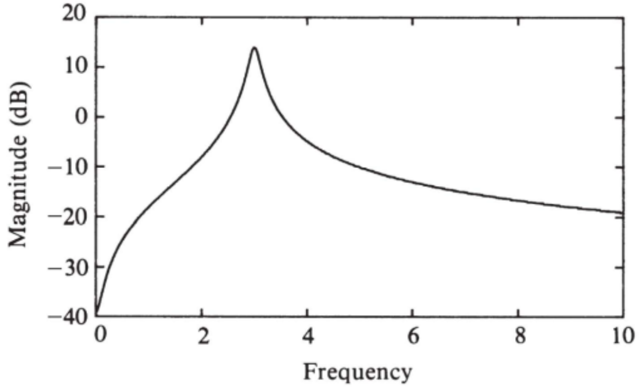
$$H(s) = \frac{1/2}{s - (-0.1 + 3j)} + \frac{1/2}{s - (-0.1 - 3j)}$$

to the z–domain (partial fractions pole mapping) yields

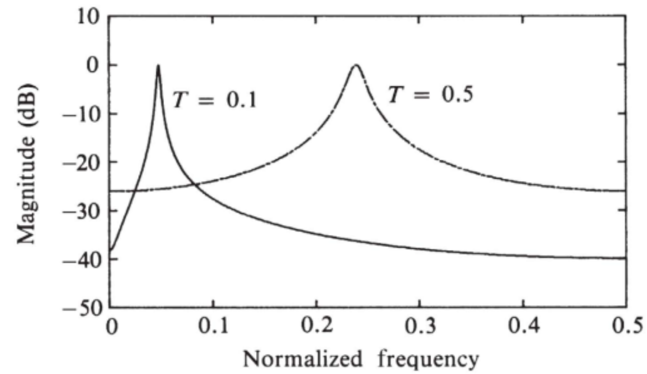
$$\begin{aligned} H(z) &= \frac{1/2}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{1/2}{1 - e^{-0.1T} e^{-j3T} z^{-1}} \\ &= \frac{1 - (e^{-0.1T} \cos 3T) z^{-1}}{1 - (2e^{-0.1T} \cos 3T) z^{-1} + e^{-0.2T} z^{-2}} \end{aligned}$$

System	Zero	Poles	Comments
$H(s)$	$s = -0.1$	$p_k = -0.1 \pm j3$	Poles in LHP $\Rightarrow$ stable
$H(z)$	$z = e^{-0.1T} \cos 3T$	$p_k = e^{-0.1T} e^{\pm j3T}$	Poles in unit circle $\Rightarrow$ stable

Result: Zero mapping is not simply $z = e^{sT}$. This and aliasing effects limit the use of impulse invariance



Frequency response of $H(s)$



Frequency response of $H(z)$

for $T = 0.1$ and 0.5

Note: T should be chosen small to avoid aliasing; Method only appropriate for lowpass and a bandpassed filters.

Summary of Impulse Invariance IIR Filter Design

1. Express $H(s)$ in partial fraction form
2. Map poles to the z–plane via $z = e^{sT}$
3. Express $H(z)$ in partial fraction form, converting to other representations as appropriate (e.g., difference equations)

IIR Filter Design by the Bilinear Transformation

Objective: Design a discrete–time IIR filter by approximating the integral (rather than the derivative, as before)

Suppose

$$H(s) = \frac{b}{s+a} \qquad \Rightarrow \dot{y}(t) + ay(t) = bx(t)$$

where $\dot{y} = \frac{dy(t)}{dt}$. Also note that

$$y(t) = \int_{-\infty}^{t} \dot{y}(\tau)d\tau = \int_{t_0}^{t} \dot{y}(\tau)d\tau + y(t_0)$$

Let $t = nT$ and $t_0 = (n-1)T$. Then (for T small)

$$\begin{aligned}\int_{t_0}^{t} \dot{y}(\tau)d\tau &\approx T\frac{(\dot{y}(t_0) + \dot{y}(t))}{2} \\ &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1))\end{aligned}$$

$$\int_{t_0}^{t} \dot{y}(\tau)d\tau \quad \rightarrow \quad \text{area under } \dot{y}(t) \text{ between } t_0 \text{ and } t$$

$$\frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) \quad \rightarrow \quad \text{average of range start \& end points} \times \text{width}$$

Result: Approximately equal for small T

Thus

$$\begin{aligned} y(t) &= \int_{t_0}^{t} \dot{y}(\tau)d\tau + y(t_0) \\ \Rightarrow y(n) &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) + y(n-1) \qquad (*) \end{aligned}$$

We need an expression for $\dot{y}(n)$ in $(*)$. Recall $H(s) = \frac{b}{s+a}$ gives

$$\begin{aligned} \dot{y}(t) + ay(t) &= bx(t) \\ \Rightarrow \dot{y}(n) &= -ay(n) + bx(n) \qquad (**) \end{aligned}$$

Substitute $(**)$ into $(*)$

$$
\begin{aligned}
y(n) &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) + y(n-1) \\
&= \frac{T}{2}([-ay(n) + bx(n)] + [-ay(n-1) + bx(n-1)]) + y(n-1)
\end{aligned}
$$

Rearranging, and then taking the z–transform, gives

$$
\begin{aligned}
\left(1 + \frac{Ta}{2}\right) y(n) - \left(1 - \frac{Ta}{2}\right) y(n-1) &= \frac{Tb}{2}(x(n) + x(n-1)) \\
Y(z)\left(1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}\right) &= X(z)\frac{bT}{2}(1 + z^{-1}) \\
\Rightarrow H(z) = \frac{Y(z)}{X(z)} &= \frac{\frac{bT}{2}(1 + z^{-1})}{1 + \frac{aT}{2} - (1 - \frac{aT}{2})z^{-1}}
\end{aligned}
$$

The denominator needs to be rearranged to lend insight

The denominator is

$$
\begin{aligned}
1+\frac{aT}{2}-(1-\frac{aT}{2})z^{-1}+\left[\frac{aT}{2}z^{-1}-\frac{aT}{2}z^{-1}\right] \\
&= 1-\frac{aT}{2}z^{-1}-(1-\frac{aT}{2})z^{-1}+\frac{aT}{2}(1+z^{-1}) \\
&= 1-z^{-1}+\frac{aT}{2}(1+z^{-1})
\end{aligned}
$$

Thus $H(z)$ is expressed as

$$
\begin{aligned}
H(z) &= \frac{b\frac{T}{2}(1+z^{-1})}{(1-z^{-1})+\frac{aT}{2}(1+z^{-1})} \\
&= \frac{b}{\frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})+a} \\
&= H(s)\big|_{s=\frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})}
\end{aligned}
$$

Result: The mapping $s=\frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})$ defines the bilinear transformation

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation for s.

Thus,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

or

$$s = \frac{2}{T}\frac{(re^{j\omega}-1)(re^{-j\omega}+1)}{(re^{j\omega}+1)(re^{-j\omega}+1)} = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

Since $s = \sigma + j\Omega$, we get

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]$$

And

$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

From the above equation for , we observe that if $r < 1$ then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$, and if $r = 1$, then $\sigma = 0$. Hence the left half of the s-plane maps into points inside the unit circle in the z-plane, the right half of the s-plane maps into points outside the unit circle in the z-plane and the imaginary axis of s-plane maps into the unit circle in the z-plane. This transformation results in a stable digital system.

Relation between analog and digital frequencies

On the imaginary axis of s-plane $\sigma = 0$ and correspondingly in the z-plane $r = 1$.

$$\Omega = \frac{2}{T}\left(\frac{2\sin\omega}{1+1+2\cos\omega}\right) = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{1+2\cos^2\omega/2-1}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently, we have $= 2\tan\frac{\Omega T}{2}$.

The above relation between analog and digital frequencies shows that the entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$. The entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of the unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = \alpha$ to 0) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$).

But as seen in Figure 1, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the nonlinearity of the arctangent function and usually known as frequency warping.

The effect of warping on the magnitude response can be explained by considering an analog filter with a number of passbands as shown in Figure 2(a). The corresponding digital filter will have same number of passbands, but with disproportionate bandwidth, as shown in Figure 2(a).

In designing digital filter using bilinear transformation, the effect of warping on amplitude response can be eliminated by prewarping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent using the equation $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$. These analog frequencies are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed, and then it is transformed to digital filter transfer function.

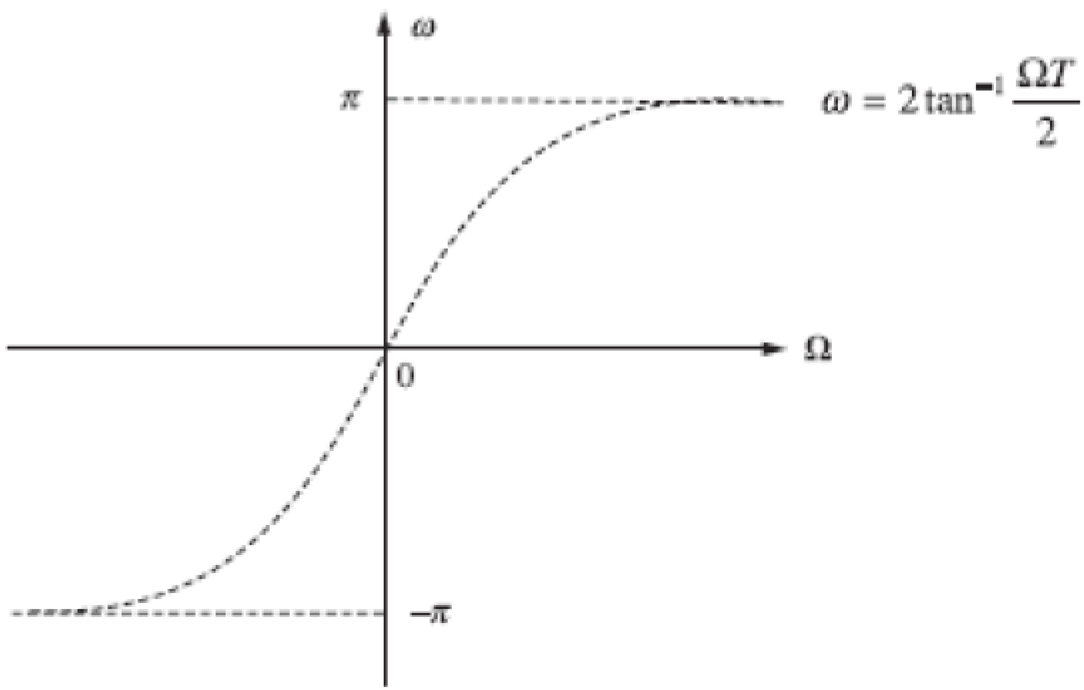


Figure 1 Mapping between Ω and ω in bilinear transformation.

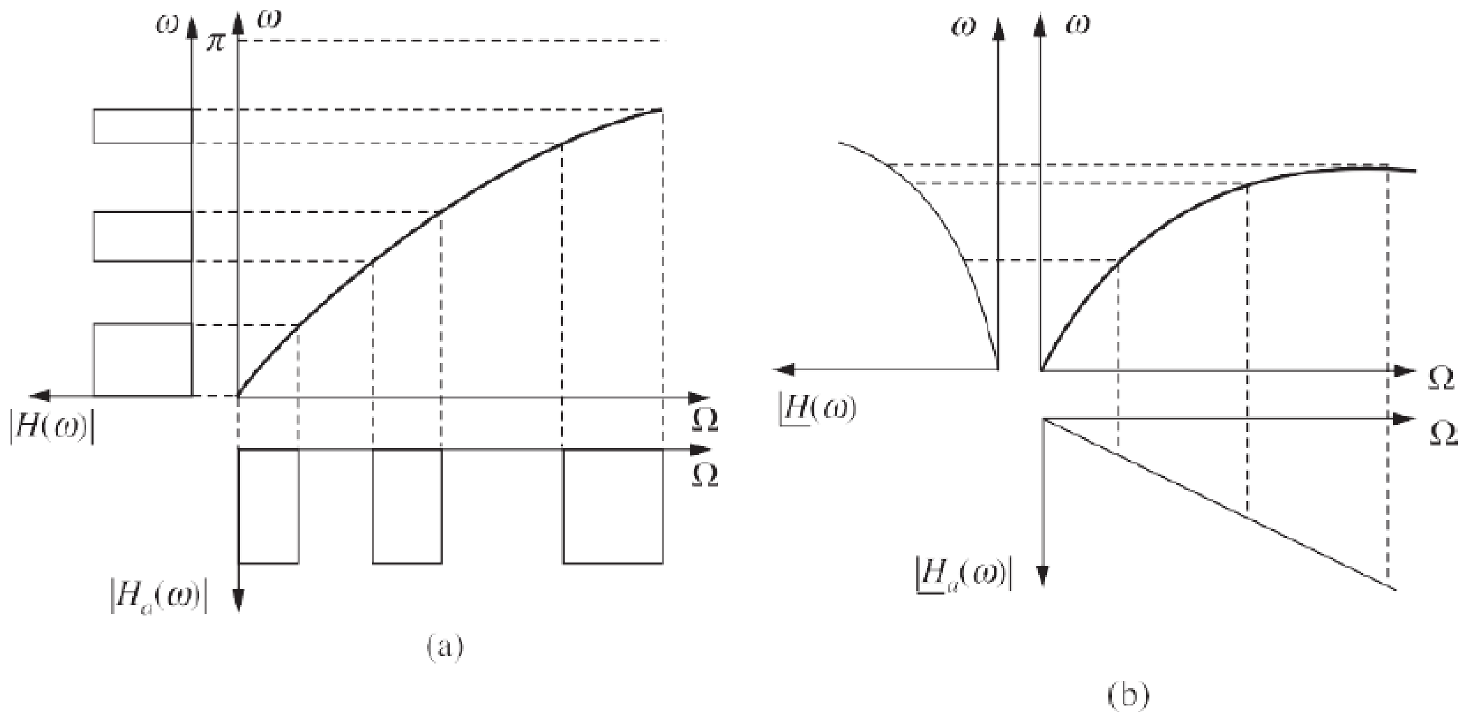


Figure 2 The warping effect on (a) magnitude response and (b) phase response.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 2(b). The phase response of corresponding digital filter will be nonlinear.

It can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore, the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant

values. A linear phase analog filter cannot be transformed into a linear phase digital filter using the bilinear transformation.

Rearranging $s = \frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})$, with $s = \sigma + j\Omega$, yields

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} = \frac{1 + \frac{\sigma T}{2} + j\Omega\frac{T}{2}}{1 - \frac{\sigma T}{2} - j\Omega\frac{T}{2}} \qquad (*)$$

Observations: From $s = \sigma + j\Omega$ and $(*)$

- $\sigma < 0 \Rightarrow |z| < 1$ (s–domain LHP maps to the inside of the z–domain unit circle – stability preserved)
- $\sigma = 0 \Rightarrow |z| = 1$ (s–plane frequency axis maps to the z–domain unit circle – frequency to frequency mapping)

Also, $\sigma = 0 \Rightarrow s = j\Omega$ and $|z| = 1$. Thus

$$\begin{aligned} s &= j\Omega = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) \\ \Rightarrow \Omega &= \frac{2}{jT}\left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right) = \frac{2}{T}\tan(\omega/2) \\ \Rightarrow \omega &= 2\arctan(\Omega T/2) \end{aligned}$$

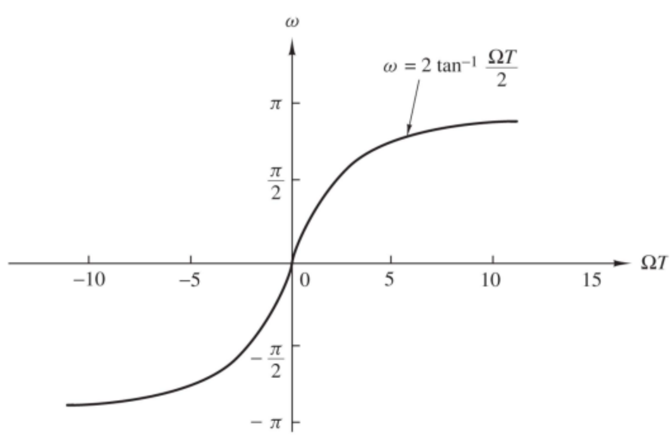
Result:

Cont. to Disc. Freq. mapping: $\omega = 2\arctan(\Omega T/2)$

Disc. to Cont. Freq. mapping: $\Omega = \frac{2}{T}\tan(\omega/2)$

This is a nonlinear frequency warping

[Cont. Freq.] $-\infty < \Omega < \infty \longleftrightarrow -\pi \leq \omega \leq \pi$ [Disc. Freq.]



Bilinear Transformation $\Omega \leftrightarrow \omega$ mapping

Bilinear Transformation Design Procedure Summary

1. Make filter specifications in the digital domain
2. Map specifications to the continuous domain
3. Use analog design techniques (look filter up)
4. Use bilinear mapping to obtain the digital filter

Example

Designed a single pole lowpass digital filter with a 3-dB bandwidth of 0.2π, using the bilinear transformation of the analog filter

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

where Ω_c is the 3-dB bandwidth of the analog filter.

First, map $\omega_c = 0.2\pi$ to the analog domain

$$\Omega_c = \frac{2}{T} \tan \omega_c/2 = \frac{2}{T} \tan 0.1\pi = 0.65/T$$

Thus for $\Omega_c = 0.65/T$

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \qquad \Rightarrow \qquad H_a(s) = \frac{0.65/T}{s + 0.65/T} \quad (*)$$

Next, apply the bilinear transformation

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

to $(*)$

$$H(z) = \frac{0.65/T}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.65/T} = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

Observations:

- T is divided out of the final $H(z)$ expression (it is a scale term, the value of which is irrelevant)
- $|H(0)| = 1$ and $|H(0.2\pi)| = 1/\sqrt{2}$, thus the 3-dB point is correct

Lecture Summary

- IIR Filter Design – Map well understood analog filters to the discrete domain (s to z plane mapping)
- Approximation of Derivatives Method for IIR Filter design –

$$z = \frac{1}{1 - sT} \qquad s = \frac{1 - z^{-1}}{T}$$

  Maps s LHP to right half of z unit circle; No highpass filters
- Impulse Invariance Method for IIR Filter design – $z = e^{sT}$; Maps s LHP to inside of z unit circle; T small to avoid aliasing; poles & zeroes have different mappings
- Bilinear Transformation Method for IIR Filter design –

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

$$\Omega = \frac{2}{T}\tan(\omega/2) \qquad \omega = 2\arctan(\Omega T/2)$$

  Maps s LHP to inside of z unit circle; No aliasing

EXAMPLE 1

Convert the following analog filter with transfer function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital IIR filter by using bilinear transformation. The digital IIR filter is having a resonant frequency of $\omega_r = \pi/2$.

Solution: From the transfer function, we observe that $\Omega_c = 3$. The sampling period T can be determined using the equation:

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\frac{\pi/2}{2} = 0.6666 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H_a(s)\Big|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{s+0.1}{(s+0.1)^2+9}\Bigg|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{3\frac{1-z^{-1}}{1+z^{-1}} + 0.1}{\left[3\frac{1-z^{-1}}{1+z^{-1}} + 0.1\right]^2 + 9}$$

$$= \frac{\left[3(1-z^{-1}) + 0.1(1+z^{-1})\right][1+z^{-1}]}{\left[3(1-z^{-1}) + 0.1\,(1+z^{-1})\right]^2 + 9(1+z^{-1})^2}$$

$$= \frac{3.1 + 0.2z^{-1} - 2.9z^{-2}}{18.61 + 0.02z^{-1} + 17.41\,z^{-2}}$$

EXAMPLE 2

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.5}{(s + 0.5)^2 + 16}$$

into a digital IIR filter using the bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \pi/2$.

Solution: From the system function, we observe that $\Omega_c = 4$. The sampling period T can be determined using the equation $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

i.e.
$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{4}\tan\frac{\pi}{4} = 0.5 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{s+0.5}{(s+0.5)^2+16}\Bigg|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.5}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.5\right]^2+16}$$

$$= \frac{\left[4(1-z^{-1})+0.5(1+z^{-1})\right][1+z^{-1}]}{\left[4(1-z^{-1})+0.5(1+z^{-1})\right]^2+16[1+z^{-1}]^2}$$

$$= \frac{4.5+z^{-1}-3.5z^{-2}}{36.25+0.5z^{-1}+28.25z^{-2}}$$

EXAMPLE 3

Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

with $T = 0.5$ s and find $H(z)$.

Solution: Given that

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

and $T = 0.5$ s

To obtain $H(z)$ using the bilinear transformation in $H_a(s)$, replace s by

$$\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{4}{(s+3)(s+4)}\bigg|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{4}{(s+3)(s+4)}\bigg|_{s=4\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{4}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+3\right]\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+4\right]}$$

$$= \frac{4}{\left[\frac{4-4z^{-1}+3+3z^{-1}}{1+z^{-1}}\right]\left[\frac{4-4z^{-1}+4+4z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4(1+z^{-1})^2}{(7-z^{-1})\,8}$$

$$= \frac{1}{2}\,\frac{(1+z^{-1})^2}{(7-z^{-1})}$$

EXAMPLE 4

Obtain $H(z)$ from $H_a(s)$ when $T = 1$ s and

$$H_a(s) = \frac{3s}{s^2 + 0.5s + 2}$$

using the bilinear transformation.

Solution: Given

$$H_a(s) = \frac{3s}{s^2 + 0.5s + 2}$$

and $T = 1$ s.

To get $H(z)$ using the bilinear transformation, put

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \text{ in } H_a(s).$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{3s}{s^2 + 0.5s + 2}\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{3 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.5\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 2}$$

$$= \frac{6\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\frac{4(1-z^{-1})^2 + (1-z^{-1})(1+z^{-1}) + 2(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$= \frac{6(1+z^{-1})}{4(1-2z^{-1}+z^{-2}) + (1-z^{-2}) + 2(1+2z^{-1}+z^{-2})}$$

$$= \frac{6+6z^{-1}}{7-4z^{-1}+5z^{-2}}$$

EXAMPLE 5

Using the bilinear transformation, obtain $H(z)$ from $H_a(s)$ when $T = 1$s and $H_a(s) = \dfrac{s^3}{(s+1)(s^2+2s+2)}$

Solution: Given that

$$H_a(s) = \frac{s^3}{(s+1)(s^2+2s+2)} \text{ and } T = 1 \text{ s.}$$

To obtain $H(z)$ using the bilinear transformation,

put $s = \dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$.

Given $T = 1$ s,

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{s^3}{(s+1)(s^2+2s+2)}\Bigg|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\left[2\frac{(1-z^{-1})}{(1+z^{-1})}\right]^3}{\left[2\frac{(1-z^{-1})}{1+z^{-1}}+1\right]\left\{\left[2\frac{(1-z^{-1})}{1+z^{-1}}\right]^2+2\left[2\frac{(1-z^{-1})}{1+z^{-1}}\right]+2\right\}}$$

$$= \frac{8(1-z^{-1})^3}{\left[2(1-z^{-1})+(1+z^{-1})\right]\left[4(1-z^{-1})^2+4(1-z^{-1})(1+z^{-1})+2(1+z^{-1})^2\right]}$$

$$= \frac{8(1-z^{-1})^3}{(3-z^{-1})[10-4z^{-1}+2z^{-2}]}$$

$$= \frac{4(1-z^{-1})^3}{(3-z^{-1})(5-2z^{-1}+2z^{-2})}$$

$$= 4\,\frac{(1-3z^{-1}+3z^{-2}-z^{-3})}{15-11z^{-1}+8z^{-2}-2z^{-3}}$$

EXAMPLE 6

A digital filter with a 3 dB bandwidth of 0.4 is to be designed from the analog filter whose system response is:

$$H(s) = \frac{\Omega_c}{s + 2\Omega_c}$$

Use the bilinear transformation and obtain $H(z)$.

Solution: We know that $\Omega_c = \frac{2}{T} \tan \frac{\omega_c}{2}$

Here the 3 dB bandwidth $\omega_c = 0.4$

$$\Omega_c = \frac{2}{T} \tan \frac{0.4\pi}{2} = \frac{1.453}{T}$$

The system response of the digital filter is given by

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\Omega_c}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2\Omega_c} = \frac{\frac{1.453}{T}}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2\left(\frac{1.453}{T}\right)}$$

$$= \frac{1.453\,(1+z^{-1})}{2(1-z^{-1}) + 2(1+z^{-1})\,1.453}$$

$$= \frac{1+z^{-1}}{3.376\; - \;0.624\,z^{-1}}$$

Design proceure for low-pass digital Butterworth IIR filter

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1

A_2 = Gain at a stopband frequency ω_2

Ω_1 = Analog frequency corresponding to ω_1

Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.

For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2},\ \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T},\quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \geq \frac{1}{2} \frac{\log \left\{ \left[\frac{1}{A_2^2} - 1 \right] \Big/ \left[\frac{1}{A_1^2} - 1 \right] \right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Step 4 Calculate the analog cutoff frequency

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1 \right]^{1/2N}}$$

Step 5 Determine the transfer function of the analog filter.
Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2 \sin \left[\frac{(2k-1)\pi}{2N} \right]$$

For normalized case, $\Omega_c = 1$ rad/s

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

Poles of normalized Butterworth filter

The Butterworth low-pass filter has a magnitude squared response given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

We know that the frequency response $H_a(\Omega)$ of an analog filter is obtained by substituting $s = j\Omega$ in the analog transfer function $H_a(s)$. Hence the system transfer function is obtained by replacing Ω by (s/j) in the above equation.

$$H_a(s)\, H_a(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}} = \frac{1}{1 + \left(\frac{s^2}{j^2\Omega_c^2}\right)^{2N}}$$

In the above equation, when s/Ω_c is replaced by Sn (i.e. Ω_c = 1 rad/s), the transfer function is called normalized transfer function.

$$H_a(s_n)H_a(-s_n) = \frac{1}{1+(-s_n^2)N}$$

n

The transfer function of the above equation will have 2*N* poles which are given by the roots of the denominator polynomial. It can be shown that the poles of the transfer function symmetrically lie on a unit circle in *s*-plane with angular spacing of π/N.

For a stable and causal filter the poles should lie on the left half of the *s*-plane. Hence the desired filter transfer function is formed by choosing the *N*-number of left half poles. When *N* is even, all the poles are complex and exist in conjugate pairs. When *N* is odd, one of the pole is real and all other poles are complex and exist as conjugate pairs. Therefore, the transfer function of Butterworth filters will be a product of second order factors.

The poles of the Butterworth polynomial lie on a circle, whose radius is ω_c. To determine the number of poles of the Butterworth filter and the angle between them we use the following rules.

- Number of Butterworth poles = 2*N*
- Angle between any two poles = 360°/(2N)

If the order of the filter N is even, then the location of the first pole is at $\theta/2$ w.r.t. the positive real axis, with the angle measured in the counter-clockwise direction. The location of the subsequent poles are respectively, at

$$\left(\frac{\theta}{2}+\theta\right),\left(\frac{\theta}{2}+2\theta\right),\left(\frac{\theta}{2}+3\theta\right),\ldots,\left(360-\frac{\theta}{2}\right)$$

If the order of the filter N is odd, then the location of the first pole is on the X-axis. The location of subsequent poles are at θ , 2θ , ..., (360 –θ) with the angle measured in the counter-clockwise direction.

If ϕ is the angle of a valid pole w.r.t. the X-axis, then the pole and its conjugate are located at $[\omega_c(\cos\phi \pm j\sin\phi)]$. .

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at œ).
2. The filter order N completely specifies the filter.
3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. At the cutoff frequency **fi**$_c$, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.

EXAMPLE 8

EXAMPLE 8

Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \le |H(\omega)| \le 1; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ s

Solution: The Butterworth digital filter is designed as per the following steps. From the given specification, we have

$$A_1 = 0.9 \text{ and } \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \text{ and } \omega_2 = \frac{3\pi}{4} \qquad \text{and } T = 1 \text{ s}$$

Step 1 Choice of the type of transformation

Here the bilinear transformation is already specified.

Step 2 Determination of the ratio of the analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

$$\frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

Step 3 Determination of the order of the filter N

$$N \geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.2)^2}-1\right]\Big/\left[\frac{1}{(0.9)^2}-1\right]\right\}}{\log 1.207}$$

$$\geq \frac{1}{2}\frac{\log\{24/0.2345\}}{\log 2.414} \geq 2.626$$

Since $N \geq 2.626$, choose $N = 3$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., –3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{2}{\left[\frac{1}{0.9^2} - 1\right]^{1/2\times 3}} = 2.5467$$

Step 5 Determination of the transfer function of the analog Butterworth filter $H_a(s)$

For odd N, we have

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For $N = 3$, we have

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

where

$$\left[\frac{(2\times 1-1)\pi}{\ }\right] = 2\sin\frac{\pi}{6} = 1$$

$$H_a(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{(2.5467)^2}{s^2+1(2.5467)\,s+(2.5467)^2}\right)$$

$$H_a(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{(2.5467)^2}{s^2+1(2.5467)\,s+(2.5467)^2}\right)$$

Step 6 Conversion of $H_a(s)$ into $H(z)$

Since bilinear transformation is to be used, the digital filter transfer function is

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \left(\frac{2.5467}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2.5467}\right)\left[\frac{(2.5467)^2}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2+2.5467\left[2\frac{1-z^{-1}}{1+z^{-1}}\right]+(2.5467)^2}\right]$$

$$= \frac{0.2332(1+z^{-1})^3}{1+0.4394z^{-1}+0.3845z^{-2}+0.0416z^{-3}}$$

EXAMPLE 9

Design a low-pass Butterworth digital filter to give response of 3 dB or less for frequencies upto 2 kHz and an attenuation of 20 dB or more beyond 4 kHz. Use the bilinear transformation technique and obtain $H(z)$ of the desired filter.

Solution: The specifications of the desired filter are given in terms of dB attenuation and frequency in Hz. First the gain is to be expressed as a numerical value and frequency in rad/s.

Here attenuation at passband frequency (ω_1) = 3 dB

Therefore, gain at passband edge frequency (ω_1) is $k_1 = -3$ dB

$$A_1 = 10^{k_1/20} = 10^{-3/20} = 0.707 = \frac{1}{\sqrt{2}}$$

Attenuation at stopband frequency (ω_2) = 20 dB

Therefore, gain at stopband edge frequency (ω_2) is $k_2 = -20$ dB

$$A_2 = 10^{k_2/20} = 10^{-20/20} = 0.1$$

Passband edge frequency = 2 kHz,

Stopband edge frequency = 4 kHz,

The design is performed as given below.
Let the sampling frequency be 10000 Hz.

$$\text{Normalized } \omega_1 = 2\pi \frac{f_1}{f_s} = 2\pi \frac{2000}{10000} = 0.4$$

$$\text{Normalized } \omega_2 = 2\pi \frac{f_2}{f_s} = 2\pi \frac{4000}{10000} = 0.8$$

Step 1 Bilinear transformation is chosen

Step 2 Ratio of analog filter edge frequencies Ω_2/Ω_1

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{T}\tan\frac{0.4\pi}{2} = 14530.8 \text{ rad/s}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{T}\tan\frac{0.8\pi}{2} = 61553.6 \text{ rad/s}$$

$$\frac{\Omega_2}{\Omega_1} = \frac{\tan\frac{\omega_2}{2}}{\tan\frac{\omega_1}{2}} = \frac{\tan 0.4\pi}{\tan 0.2\pi} = 4.236$$

Step 3 Order of the filter

$$N \geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.1)^2}-1\right]\Big/\left[\frac{1}{(1/\sqrt{2})^2}-1\right]\right\}}{\log 4.236}$$

$$\geq \frac{1}{2}\frac{\log[99/1]}{\log 4.236} \geq 1.59$$

$$N = 2$$

Step 4 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{1.4530}{\left[\frac{1}{(1/\sqrt{2})^2}-1\right]^{1/2\times2}} = 1.4530$$

Unnormalized

$$\Omega_c = f_s \times 1.4530 = 14530 \text{ rad/s}$$

Step 5 Transfer function $H_a(s)$

$$\text{For } N = 2,\ H_a(s) = \frac{\Omega_c^2}{s^2 + b_1\Omega_c s + \Omega_c^2}$$

$$\text{where } b_1 = 2\sin\left[\frac{(2\times1-1)\pi}{2\times2}\right] = 2\sin\frac{\pi}{4} = 1.414$$

$$\therefore \quad H_a(s) = \frac{(14530)^2}{s^2 + 1.414\times14530s + (14530)^2}$$

$$= \frac{2.1112\times10^8}{s^2 + 20545.42s + 2.1112\times10^8}$$

For $N = 2,\ H_a(s) = \dfrac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$

where $b_1 = 2\sin\left[\dfrac{(2\times 1-1)\pi}{2\times 2}\right] = 2\sin\dfrac{\pi}{4} = 1.414$

$$\therefore \qquad H_a(s) = \frac{(14530)^2}{s^2 + 1.414 \times 14530\, s + (14530)^2}$$

$$= \frac{2.1112 \times 10^8}{s^2 + 20545.42 s + 2.1112 \times 10^8}$$

Step 6 Conversion of $H_a(s)$ into $H(z)$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=20000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{2.112 \times 10^8}{\left[20 \times 10^3\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 2.0545 \times 10^4 \times 20 \times 10^3\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right] + 2.112 \times 10^8}$$

$$= \frac{0.528}{2.5552 - 0.946z^{-1} + 0.5008z^{-2}}$$